ON THE COMPARISION OF CERTAIN CLASSES OF BALANCED 2^8 FRACTIONAL FACTORIAL DESIGNS OF RESOLUTION V, WITH RESPECT TO THE TRACE CRITERION

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Introduction

Let T be a 2^8 fractional factorial design of resolution V, and V_T the $(\nu \times \nu)$ variance-covariance matrix of the estimates associated with T. Then T is said to be balanced if and only if the quantities $\mathrm{Var}(\hat{A}_i)$, $\mathrm{Var}(\hat{A}_{ij})$, $\mathrm{Cov}(\hat{\mu}, \hat{A}_i)$, $\mathrm{Cov}(\hat{\mu}, \hat{A}_{ij})$, $\mathrm{Cov}(\hat{A}_i, \hat{A}_j)$, $\mathrm{Cov}(\hat{A}_i, \hat{A}_{ij})$, $\mathrm{Cov}(\hat{A}_i, \hat{A}_{jk})$, $\mathrm{Cov}(\hat{A}_{ij}, \hat{A}_{ik})$ and $\mathrm{Cov}(\hat{A}_{ij}, \hat{A}_{kl})$ are all independent of i, j, k, l, where i, j, k, l are distinct integers chosen out of the set of integers $\{1, 2, \ldots, 8\}$. Thus, for example, we must have $\mathrm{Var}(\hat{A}_1) = \mathrm{Var}(\hat{A}_2)$, $\mathrm{Cov}(\hat{A}_{12}, \hat{A}_{23}) = \mathrm{Cov}(\hat{A}_{45}, \hat{A}_{47})$, etc., though not necessarily $V(\hat{A}_1) = \mathrm{Var}(\hat{A}_{12})$ or $\mathrm{Cov}(\hat{A}_{12}, \hat{A}_{13}) = \mathrm{Cov}(\hat{A}_{12}, \hat{A}_{34})$ etc.

We shall denote treatment-combinations (or runs or assemblies) by column vectors $(k_1, k_2, \ldots, k_8)$, where k_i $(=0$ or $1)$ denotes the level of the ith factor. Then any 2^8 fractional factorial design T having N runs can be represented by an $(8 \times N)$ matrix (or array) with element 0 or 1, whose columns denote the various treatment-combinations in T. It can be shown [Chakravarti (1956), Srivastava (1961, 1970)] that a necessary and sufficient condition for T to be balanced is that every $(4 \times N)$ submatrix T_0 of T should have the following property : If $\underline{v}$ is a column vector having $(4-i)$ zeros and i 1's in it (for $i = 0, 1, 2, 3, 4$), then $\underline{v}$ occurs as a column of T_0 exactly μ_i $(\geqslant 0)$ times, where μ_i is independent of $\underline{v}$ and T_0, so long as T_0 is a $(4 \times N)$ submatrix of T and $\underline{v}$ (4×1) has i 1's in it. The 5-plet $\underline{\mu}' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$ is called the "index set" of the balanced design (also called balanced array). It may be noted that the identity

$$N = \mu_0 + 4\mu_1 + 6\mu_2 + 4\mu_3 + \mu_4 \qquad (1)$$

is always satisfied. For brevity, we shall not discuss here the properties of balanced or optimal designs. For this, the interested reader is referred to Srivastava and Chopra (1971*a*, *b*, *c*). where a discussion of previous work and interrelations with other areas will be found. Here, it may be pointed out that if T is a balanced resolution V design, then the variance matrix V_T is a function of $\mu_0, \mu_1, \mu_2, \mu_3$ and μ_4 alone. The following theorem, a special case of a result for general 2^m factorials proved in Srivastava and Chopra (1971*a*), will be needed in the sequel.

Theorem 1. Let T be a balanced 2^8 fractional factorial design of resolution V with N runs, and index set $\underline{\mu}' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$. Define γ_i (i=1, 2, 3, 4, 5) by

$$\gamma_1 = N = \mu_0 + 4\mu_1 + 6\mu_2 + 4\mu_3 + \mu_4 \tag{2}$$

$$\gamma_2 = (\mu_4 - \mu_0) + 2(\mu_3 - \mu_1) \tag{3}$$

$$\gamma_3 = \mu_4 - 2\mu_2 + \mu_0 \tag{4}$$

$$\gamma_4 = (\mu_4 - \mu_0) - 2(\mu_3 - \mu_1) \tag{5}$$

$$\gamma_5 = \mu_0 - 4\mu_1 + 6\mu_2 - 4\mu_3 + \mu_4 \tag{6}$$

Then (a) the following inequalities (7)—(12) must hold.

$$\gamma_1 - 2\gamma_3 + \gamma_5 \geqslant 0, \text{ or equivalently } \mu_2 \geqslant 0 \tag{7}$$

$$c_4 = 2\gamma_1 + 3\gamma_3 - 5\gamma_5 \geqslant 0, \text{ or equivalently } \mu_1 + \mu_3 \geqslant \frac{6}{7}\mu_2 \tag{8}$$

$$c_5 \equiv (\gamma_1 - \gamma_3)[\gamma_1 - 5\gamma_5 + 4\gamma_3] - 6(\gamma_2 - \gamma_4)^2 > 0, \text{ or equivalently } 4\mu_2^2 < \mu_2(\mu_1 + \mu_3) + 6\mu_1\mu_3 \tag{9}$$

$$3\gamma_1 + 19\gamma_3 + 15\gamma_5 \geqslant 0. \tag{10}$$

$$c_2 \equiv 3\gamma_1^2 - 22\gamma_2^2 + 56\gamma_3^2 - 126\gamma_4^2 + 38\gamma_1\gamma_3 + 30\gamma_1\gamma_5 - 84\gamma_2\gamma_4 + 105\gamma_3\gamma_5 \geqslant 0. \tag{11}$$

$$c_3 \equiv \gamma_1^3 - 196\gamma_3^3 + 19\gamma_1^2\gamma_3 + 15\gamma_1^2\gamma_5 + 56\gamma_1\gamma_3^2 + 105\gamma_1\gamma_3\gamma_5 - 22\gamma_1\gamma_2^2 - 126\gamma_1\gamma_4^2 - 84\gamma_1\gamma_2\gamma_4 + 16\gamma_2^2\gamma_3 - 120\gamma_2^2\gamma_5 + 336\gamma_2\gamma_3\gamma_4 > 0. \tag{12}$$

(b) We must have

$$\text{tr } V_T = \frac{c_2}{c_3} + \frac{7c_4}{c_5} + \frac{5}{4\mu_2}. \tag{13}$$

Now, suppose for a given N, one desires to find a balanced design T such that tr V_T is a minimum. Then one would solve the diophantine equation (1), obtain, various solutions $(\mu_0, \mu_1, \mu_2, \mu_1, \mu_0)$, substitute these in (13), and obtain the solution [say, $(\mu_0^*, \mu_1^*, \mu_2^*, \mu_3^*, \mu_4^*) \equiv \underline{\mu}^{*\prime}$] for which the expression on the r.h.s. of (13) is minimized. Note that this does not solve the problem of obtaining a balanced design which is optimal with respect to the trace criterion. The reason is that after obtaining $\underline{\mu}^{*\prime}$, one has to consider the (usually very non trivial) combinatorial-mathematical problem of determining whether there exists a balanced design T^* with index $\underline{\mu}^{*\prime}$. Conditions (7)–(12), and other classes of conditions developed in Srivastva and Chopra (1971c) and elsewhere, are helpful for this purpose, However, very often the problem of finding $\underline{\mu}^{*\prime}$ itself is quite cumbersome. The reason is that given N, equation (1) has usually a very large number of possible solutions $\underline{\mu}'$. For example, when $N=48$, there are thousands of possible solutions. Substitution of all these solutions in (13) and the subsequent comparing of the value of the trace is quite cumbersome.

One therefore needs to find results using which a lot of such solutions $\underline{\mu}'$ could be readily screened out of consideration. In this paper, we prove a couple of theorems, which are very useful in the last-mentioned sense.

2. Comparison of index sets under the trace-criterion.

Balanced designs T with index sets $(\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$ such that $\mu_0=\mu_4$ and $\mu_1=\mu_3$, have been called (1, 0) symmetric in Srivastava (1970). In this section, we show that for 2^8 factorials, such designs [i.e. those having index sets of the form $(\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$] gives rise to a lower value of tr V_T than designs with index sets of the form $(\mu_0-z, \mu_1, \mu_2, \mu_1, \mu_0+z)$ or $(\mu_0, \mu_1-z, \mu_2, \mu_1+z, \mu_0)$ with $z\neq0$. We conjecture that such a result is true for general 2^m factorials, but the proof of this is still an open problem. Indeed, an even more general result is perhaps true, namely that an index set of the form $(\mu_0, \mu_1, \mu_2, \mu_1, \mu_1)$ would be 'better than' one of the form $(\mu_0-z, \mu_1-z', \mu_2, \mu_1+z', \mu_0+z)$, where both z and z' are nonzero.

Theorem 2. If $T_1(\mu_0, \mu_1, \mu_2, \mu_1, \mu_0)$ and $T_2(\mu_0-z, \mu_1, \mu_2, \mu_1, \mu_0+z)$, $z\neq0$, be balanced resolution V designs of the 2^8 type, whose parameters satisfy (7)–(12), then $\text{tr}(V_{T_1})<\text{tr}(V_{T_2})$.

Proof. If possible, let us assume $\text{tr}(V_{T_1})\geqslant\text{tr}(V_{T_2})$. Let $\mu_0=x\mu_2$, $\mu_1=y\mu_2$, where $x, y>0$ for either x or $y=0$ will make T_1 a singular design. (A design will be called singular if all the ν parameters are not estimable).

For the array T_1, using (2)–(6), we have

$$\gamma_1=2\mu_2(x+4y+3),\ \gamma_2=\mu_4=0,$$
$$\gamma_3=2\mu_2(x-1) \text{ and } \gamma_5=2\mu_2(x-4y+3) \tag{14}$$

Using (14) and (7)–(12), we have respectively.

$$y>3/7 \tag{15}$$
$$y>2/3 \tag{16}$$
$$37x+35>48y \tag{17}$$
$$a\equiv58x^2-108y^2-61xy+93x+85y-19>0. \tag{18}$$
$$b\equiv-28y^3+14x^2y-49xy^2+44x^2+9xy+25y^2-2x+13y-10>0. \tag{19}$$

If we take equality in (18), then the curve it represents is a hyperbola. On the other hand, equality in (19) implies that the curve it represents has two branches (considered as an equation in x). It can be shown that there is no point common to these two branches for any $y>2/3$ [condition (16)]. In order to see this, it can be easily checked that the equation

$$(-49y^2+9y-2)^2-4(14y+44)(-28y^3+25y^2+13y-10)=0$$

does not have any root for $y>2/3$.

For the balanced design T_2, we have

$$\gamma_1=2\mu_2(x+4y+3),\ \gamma_2=2z=\gamma_4 \tag{20}$$

$$\gamma_3=2\mu_2(x-1) \text{ and } \gamma_5=2\mu_2(x-4y+3)$$

These together with (7)–(12), give us respectively (15)–(17) and

$$\frac{\mu_2^{\,2}}{z^2}>\frac{58}{a} \tag{21}$$

$$\frac{\mu_2^{\,2}}{z^2}>\frac{14y+44}{b} \tag{22}$$

Hence, for both T_1 and T_2, (15) and (16) imply $y>2/3$. Also $tr\, V_{T_1} \geqslant tr V_{T_2}$ implies $\frac{a}{b}\geqslant\frac{a-928z^2}{b-z^2(928\gamma_1-1408\gamma_3+480\gamma_5)}$, since c_4 and c_5 are the same for both T_1 and T_2, in view of (14), (20) and theorem 1.3. Hence, from (18), (19), (21) and (22), we have (since both denominators are positive in the last inequality) $a(928\gamma_1-1408\gamma_3+480\gamma_5)\leqslant 928b$. This, after some simplification, is reduced to

$$28y^3-1253y^2+497xy^2-476xy+1052x+680y-64\leqslant 0. \tag{23}$$

In order to complete the proof of the theorem, we show that there does not exist any point (x, y) where (15)–(19) and (23) are simultaneously satisfied. First of all, we solve the inequalities (15)–(19) and (23) in terms of x. We have $y>2/3$, $x>\frac{48y-35}{37}=f_1(y)$, say. Also

$$x>\frac{(61y-93)+\sqrt{(61y-93)^2-232(-108y^2+85y-19)}}{116}=f_2(y), \text{ say };$$

or

$$x<\frac{(61y-93)-\sqrt{(61y-93)^2-232(-108y^2+85y-19)}}{116}=f_2{}^*(y), \text{ say };$$

$$x>\frac{28y^2+3y-10}{7y+22}=f_3(y), \text{ say };\ x<\frac{-14y^2-30y+44}{4(7y+22)}=f_3{}^*(y), \text{ say };$$

$$x\leqslant\frac{1253y^2-28y^3-680y+64}{497y^2-476y+1052}=f_4(y), \text{ say}.$$

Since x has to be > 0 and clearly $f_2^*(y)<0$ for $y> 2/3$, therefore we do not consider $x<f_2^*(y)$. Thus we are to prove the theorem in the following regions :

$$E_1 : y>\frac{2}{3},\ x>f_1(y),\ x>f_2(y),\ x>f_3(y),\ x\leqslant f_4(y)$$

$$E_2 : y>\frac{2}{3},\ x>f_1(y),\ x>f_2(y),\ x<f_3^*(y), \text{ and } x\leqslant f_4(y)$$

First, we take the region E_1 and show that amongst $f_1(y)$, $f_2(y)$ and $f_3(y)$ the dominant condition is the one concerning $f_3(y)$. (*i*) Consider $\phi_1(y)=f_2(y)-f_1(y)$. Clearly $\phi_1(1)>0$. From $\phi_1(y)=0$, we have $(28{,}432{,}992)\ y^2-(38{,}430{,}336)\ y+(17{,}491{,}872)=0$, which is observed to have imaginary roots. Thus, because of the continuity of ϕ_1, $\phi_1(y)>0$, in the region $y>\frac{2}{3}$. Hence $f_2(y)>f_1(y)$. (*ii*) Next, we take $\phi_2(y)=f_3(y)-f_2(y)$. We have $\phi_2(1)>0$. Put $\phi_2(y)=0$, we have the quartic $168y^4-238y^3+3y^2+267y-142=0$. It can be checked that two of its real roots are $-1, \frac{2}{3}$ while the remaining two are imaginary. (*iii*) Finally, we take $\phi_3(y)=f_3(y)-f_4(y)$. Now $\phi_3(1)>0$. Also, $\phi_3(y)=0$ is observed to have two real roots $y=-1$, $y=\frac{2}{3}$. The reduced equation is $56y^2-98y+71=0$ which has imaginary roots. This proves that there does not exist any point (x, y) contained in the region E_1.

Next, we take the region E_2. In order to complete the proof for E_2, we shall prove that $\{x>f_2(y)\}\cap\{x<f_3^*(y)\}$ is an empty set. It means that we are to show $f_2(y)-f_3^*(y)>0$. This is true if and only if $(7y+22)^2\ \{(61y-93)^2+232\ (108y^2-85y+119)\}>(3322-1561y-833y^2)^2$, which, after some simplification, gives us

$716{,}184y^4+4{,}740{,}456y^3+8{,}097{,}264y^2-643{,}104y-4{,}716{,}096>0$ which is obviously true. Hence the proof of the theorem is complete.

Theorem 3. If $T_1(\mu_0, \mu_1, \mu_2, \mu_1, \mu_0)$ and $T_2\ (\mu_0, \mu_1-z, \mu_2, \mu_1+z, \mu_0)$, $z>0$, are resolution V designs whose parameters satisfy (7)–(12), then $tr\ V_{T_1}<tr\ V_{T_2}$.

Proof. If possible, let us suppose that $tr\ V_{T_1}\geqslant tr\ V_{T_2}$ and, as before, set $\mu_0=x\mu_2$, $\mu_1=y\mu_2$, where $x, y>0$. For T_1, we have, the γ's are given by (14). Furthermore, (15)–(19) of theorem 2 hold here also. For T_2, γ_1, γ_3 and γ_5 remain same, as before, and

$$\gamma_2=4z=-\gamma_4. \tag{24}$$

From (7)–(12), we obtain

$$y>\frac{3}{7} \tag{25a}$$

$$\frac{\mu_2^2}{z^2}>\frac{3}{3y^2+y-2} \tag{25b}$$

$$37x+35>48y \tag{25c}$$

$$\frac{\mu_2^2}{z^2}>\frac{64}{a}, \text{ where '}a\text{' is as given in (18).} \tag{25d}$$

$$\frac{\mu_2^2}{z^2}>\frac{63x-28y+29}{b}, \text{ where '}b\text{' is given in (19).} \tag{25e}$$

Next, $tr\ V_{T_1} \geqslant tr\ V_{T_2}$ implies

$$\frac{c}{d}+\frac{e}{f}\geqslant\frac{c-1024\,z^2}{d-z^2(1024\gamma_1+5120\gamma_3+1920\gamma_5)}+\frac{e}{f-384z^2}, \tag{26}$$

where $c=3\gamma_1^2+38\gamma_1\gamma_3+30\gamma_1\gamma_5+56\gamma_3^2+105\gamma_3\gamma_5$

$d=\gamma_1^3-196\gamma_3^3+56\gamma_1\gamma_3^2+19\gamma^2{}_1\gamma_3+15\gamma_1^2\gamma_5+105\gamma_1\gamma_3\gamma_5$

$e=7(2\gamma_1-5\gamma_5+3\gamma_3)=56\mu_2(7y-3)$

$f=(\gamma_1-5\gamma_5+4\gamma_3)(\gamma_1-\gamma_3)=128\mu_2^2(3y-2)(y+1)$

in order to prove the theorem, we shall prove first of all that

$$\frac{63x-28y+29}{b}>\frac{64}{a}>\frac{3}{3y^2+y-2} \tag{26a}$$

We take the last two,

$$\frac{64}{58x^2-108y^2-61xy+93x+85y-19}>\frac{3}{3y^2+y-2}$$

Now

$$\frac{64}{58x^2-108y^2-61xy+93x+85y-19}-\frac{3}{3y^2+y-2}$$
$$=-\frac{174x^2-516y^2-183xy+279x+191y+71}{\text{denominator}}.$$

In order to prove the result, we shall prove that $174x^2-516y^2-183xy+279x+191y+71$ is negative. Now $174x^2-516y^2-183xy+279x+191y+71$ $=174x^2+x(279-183y)+(-516y^2+191y+71)$. Regarding it as a quadratic in x, we have 'Discriminant' $=75(5235y^2-3134y+379)$ which is always positive for any $y>\frac{2}{3}$ and this proves the result.

Next, $\frac{63x-28y+29}{b}>\frac{64}{a}$, after some simplification, reduces to

$$f(x, y)\equiv3654x^3+4816y^3-6363x^2y-1960xy^2+4725x^2-7112y^2+406xy+1628x+2165y+89>0. \tag{27}$$

From $\gamma_1=2\mu_0+8\mu_1+6\mu_2$, we have

$$x+4y=\frac{\gamma_1}{2\mu_2}-3=l \text{ (say), where } l=\frac{\gamma_1}{2\mu_2}-3>0. \tag{28}$$

Substituting $x=l-4y$, from (28), in the l.h.s. of (27), we have

$$f(l, y)=3654l^3+4725l^2+224{,}336ly^2+1628l+67{,}864y^2+89-(11{,}211l^2y+37{,}394ly+323{,}008y^3+4347)$$

It is obvious from (28) that $l/4 \geqslant y$ and hence $2/3<y \leqslant l/4$. Substituting $y=al$ in $f(l, y)$, where $a \leqslant 1/4$, we have

$$f(l, al)=l^3(-323{,}008a^3+224{,}336a^2-11{,}211a+3{,}654)+l^2(67{,}864a^2-37{,}394a+4725)+l(-4347a+1628)+89.$$

It can be easily checked that each of the coefficients of l^3, l^2, l are positive for $a \leqslant 1/4$. Thus $f(l, y)>0$, for $l>0$ and $y \leqslant l/4$. This proves (26a), which we write now as

$$\frac{\mu_2{}^2}{z^2}>\frac{63x-28y+29}{b}>\frac{64}{a}>\frac{3}{(3y^2+y-2)} \tag{29}$$

From (26), after some simplification, we obtain

$$z^2[3(y+1)(3y-2)\{a(63x-28y+29)-64b\}+21(7y-3)(63x-28y+29)b] \geqslant \mu_2{}^2[21(7y-3)b^2+(y+1)^2(3y-2)^2\{a(63x-28y+29)-64b\}] \tag{30}$$

The inequality in (28) does not change if, after dividing both sides by z^2, we replace $\frac{\mu_2{}^2}{z^2}$ by $\frac{(63x-28y+29)}{b}$ in the right hand side. Thus $3(y+1)(3y-2)[a(63x-38y+29)-64b]+21b(7y-3)(63x-28y+29) \geqslant \frac{(63x-28y+29)}{b}$ $[21(7y-3)b^2+(y+1)^2(3y-2)^2\{a(63x-28y+29)-64b\}]$. Multiplying both sides by b, $b>0$, and simplifying, we get

$$3(y+1)(3y-2)b[a(63x-28y+29)-64b] \geqslant$$
$$(y+1)^2\ 3(y-2)^2\ (63x-28y+29)[a(63x-28y+29)-64b].$$

Using (27) in above, we find that (28) will hold if

$3b \geqslant (y+1)(3y-2)(63x-28y+29)$, which gives us a contradiction. This completes the proof of the theorem.

3. Summary

Consider fractional factorial designs of the 2^8 experiment. For a given N (the number of runs), let T_1 and T_2 be two competing 2^8 factorial designs with the same value of N. Furthermore, suppose both T_1 and T_2 are of resolution V, *i.e.* the general mean μ, the main effects A_i, and the 2-factor interaction A_{ij} are estimable, assuming the 3-factor and higher order interactions to be negligible. Note that the number of parameters to be estimated is v, where $v=1+8+8(8-1)/2=37$. Let V_{T_i} $(i=1, 2)$ be the $(v \times v)$ variance-covariance matrix of the estimates of the parameters, when the design T_i is used. Finally, assume further that both T_1 and T_2 are "balanced" (see the definition in the introduction) with "index sets" of the form $(\mu_0, \mu_1, \mu_2, \mu_1, \mu_0)$ and $(\mu_0-\eta, \mu_1-\eta', \mu_2, \mu_1+\eta', \mu_0+\eta)$ respectively, were $\mu_i \geqslant 0$ $(i=0, 1, 2)$, $|\eta'| \leqslant \mu_1$, $|\eta| \leqslant \mu_0$, and either η or η' (but not both) are zero. Then, in this paper, we prove that $tr\ V_{T_1} \leqslant tr\ V_{T_2}$; in other words T_1 is at least as good as T_2 under the trace criterion.

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